

Running coupling constant and masses in QCD, the meson spectrum

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Abstract. In line with some previous works, we study in this paper the meson spectrum in the framework of a second order quark-antiquark Bethe-Salpeter formalism which includes confinement. An analytic one loop running coupling constant $\alpha_s(Q)$, as proposed by Shirkov and Sovlovtssov, is used in the calculations. As for the quark masses, the case of a purely phenomenological running mass for the light quarks in terms of the c. m. momentum is further investigated. Alternatively a more fundamental expression $m_P(Q)$ is introduced for light and strange quarks, combining renormalization group and analyticity requirements with an approximate solution of the Dyson-Schwinger equation. The use of such running coupling constant and masses turns out to be essential for a correct reproduction of the the light pseudoscalar mesons.

1. INTRODUCTION

In a series of papers [1, 2] we have applied a second order Bethe-Salpeter formalism¹, previously established [3], to the evaluation of the quark-antiquark spectrum, in the context of QCD. Taking advantage of a Feynman-Schwinger representation for the quark propagator in an external field, the kernels of the Bethe-Salpeter and the Dyson-Schwinger equations were obtained, starting from an appropriate ansatz on the Wilson loop correlator. Such an ansatz consisted in adding an area term to the lowest perturbative expression of $\ln W$. By a 3D reduction of the original 4D BS equation a mass operator was obtained and applied to the determination of the $q\bar{q}$ bound states [4].

In that way, using a fixed strong coupling constant α_s and appropriate values for the other variables, the entire spectrum was reasonably well reproduced with, however, the relevant exception of the light pseudoscalar mesons (π, K, η_s). Agreement even for the latter states could be obtained using an analytic running coupling constant $\alpha_s(Q)$ proposed by Shirkov and Sovlovtssov, which is modified in the infrared region with respect to the ordinary purely perturbative expression [5, 6]. In conjunction it was also necessary to use a phenomenological running constituent mass for the light quarks u and d , written as a polynomial in the center of the mass quark momentum \mathbf{k} [2].

In this paper we reconsider and improve the above procedure from two aspects:

a) we evaluate the hyperfine $^3S_1 - ^1S_0$ separation for the light pseudoscalar mesons to the second rather than to the first order perturbation theory,

¹ second order in the sense of the differential equations

b) we use running constituent masses for u, d and s quarks, obtained by an approximate solution of the appropriate DS equations and analytic running current masses.

As a consequence of a) a significant improvement is obtained in agreement with the data already with a phenomenological running mass for the light quarks. As for case b), preliminary calculations seem to provide results numerically similar to the above ones, but more satisfactory from the conceptual point of view.

The plan of the remaining part of the paper is as follows. In Sect. 2 we briefly recall the second order BS formalism to establish notations. In Sect. 3 we discuss the DS equation and the 3D reduction of the BS equation. In Sect. 4 we consider the infrared behavior of the running coupling constant and obtain the corresponding running masses. In sect. 5 and 6 we report our results and draw some conclusions.

2. SECOND ORDER BETHE-SALPETER FORMALISM

In the QCD framework a *second order* four point quark-antiquark function and a full quark propagator can be defined as

$$H^{(4)}(x_1, x_2; y_1, y_2) = -\frac{1}{3} \text{Tr}_{\text{color}} \langle \Delta_1(x_1, y_1; A) \Delta_2(y_2, x_2; A) \rangle \quad (1)$$

and

$$H^{(2)}(x - y) = \frac{i}{\sqrt{3}} \text{Tr}_{\text{color}} \langle \Delta(x, y; A) \rangle, \quad (2)$$

where

$$\langle f[A] \rangle = \int DA M_F[A] e^{iS_G[A]} f[A], \quad (3)$$

$M_F[A] = \text{Det} \Pi_{j=1}^2 [1 + g \gamma^\mu A_\mu (i \gamma_j^\nu \partial_{j\nu} - m_j)^{-1}]$ and $\Delta(x, y; A)$ is the *second order* quark propagator in an external gauge field.

The quantity Δ is defined by the second order differential equation

$$(D_\mu D^\mu + m^2 - \frac{1}{2} g \sigma^{\mu\nu} F_{\mu\nu}) \Delta(x, y; A) = -\delta^4(x - y), \quad (4)$$

($\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and $D_\mu = \partial_\mu + ig A_\mu$) and it is related to the corresponding first order propagator by $S(x, y; A) = (i \gamma^\nu D_\nu + m) \Delta(x, y; A)$.

The advantage of considering second order quantities is that the spin terms are more clearly separated and it is possible to write for Δ a generalized Feynman-Schwinger representation, *i. e.* to solve eq. (4) in terms of a quark path integral [3, 1]. Using the latter in (1) or (2) a similar representation can be obtained for $H^{(4)}$ and $H^{(2)}$.

The interesting aspect of this representation is that the gauge field appears only through a Wilson line correlator W . In the limit $x_2 \rightarrow x_1$, $y_2 \rightarrow y_1$ or $y \rightarrow x$ the Wilson lines close in a single Wilson loop Γ and if Γ stays on a plane, $i \ln W$ can be written in a first approximation as the sum of its lowest perturbative expression and an area term

$$i \ln W = \frac{4}{3} g^2 \oint dz^\mu \oint dz^{\nu'} D_{\mu\nu}(z - z') + \quad (5)$$

$$\sigma \oint dz^0 \oint dz^{0'} \delta(z^0 - z^{0'}) |\mathbf{z} - \mathbf{z}'| \int_0^1 d\lambda \left\{ 1 - \left[\lambda \frac{d\mathbf{z}_\perp}{dz^0} + (1 - \lambda) \frac{d\mathbf{z}'_\perp}{dz^{0'}} \right]^2 \right\}^{\frac{1}{2}}.$$

The area term here is written as the algebraic sum of successive equal time strips and $d\mathbf{z}_\perp = d\mathbf{z} - (d\mathbf{z} \cdot \mathbf{r})\mathbf{r}/r^2$ denotes the transversal component of $d\mathbf{z}$. The basic assumption now is that in the center of mass frame (5) remains a good approximation even in the general case, *i. e.* for non flat curves and when $x_2 \neq x_1$, $y_2 \neq y_1$ or $y \neq x$.

Then, by appropriate manipulations on the resulting expressions, an inhomogeneous Bethe-Salpeter equation for the 4-point function $H^{(4)}(x_1, x_2; y_1, y_2)$ and a Dyson-Schwinger equation for $H^{(2)}(x - y)$ can be derived in a kind of generalized ladder and rainbow approximation. This should appear plausible, even from the point of view of graph resummation, for the analogy between the perturbative and the confinement terms in (5). We may refer to such terms as a *gluon exchange* and a *string connection*.

In momentum representation, the corresponding homogeneous BS-equation becomes

$$\begin{aligned} \Phi_P(k) = & -i \int \frac{d^4 u}{(2\pi)^4} \hat{I}_{ab} \left(k - u; \frac{1}{2}P + \frac{k+u}{2}, \frac{1}{2}P - \frac{k+u}{2} \right) \\ & \hat{H}_1^{(2)} \left(\frac{1}{2}P + k \right) \sigma^a \Phi_P(u) \sigma^b \hat{H}_2^{(2)} \left(-\frac{1}{2}P + k \right), \end{aligned} \quad (6)$$

where we have set $\sigma^0 = 1$; $a, b = 0, \mu\nu$; the center of mass frame has to be understood, $P = (m_B, \mathbf{0})$; $\Phi_P(k)$ denotes an appropriate *second order* wave function².

Similarly, in terms of the irreducible self-energy, defined by $\hat{H}^{(2)}(k) = \frac{i}{k^2 - m^2} + \frac{i}{k^2 - m^2} i\Gamma(k) \hat{H}^{(2)}(k)$, the DS-equation can be written

$$\hat{\Gamma}(k) = \int \frac{d^4 l}{(2\pi)^4} \hat{I}_{ab} \left(k - l; \frac{k+l}{2}, \frac{k+l}{2} \right) \sigma^a \hat{H}^{(2)}(l) \sigma^b. \quad (7)$$

The kernels in (6) and (7) are the same in the two equations, consistently with the requirement of chiral symmetry limit [7], and are given by

$$\begin{aligned} \hat{I}_{0;0}(Q; p, p') &= 16\pi \frac{4}{3} \alpha_s p^\alpha p'^\beta \hat{D}_{\alpha\beta}(Q) + \\ &+ 4\sigma \int d^3 \zeta e^{-i\mathbf{Q} \cdot \zeta} |\zeta| \varepsilon(p_0) \varepsilon(p'_0) \int_0^1 d\lambda \{ p_0^2 p_0'^2 - [\lambda p_0' \mathbf{p}_T + (1 - \lambda) p_0 \mathbf{p}_T']^2 \}^{\frac{1}{2}} \\ \hat{I}_{\mu\nu;0}(Q; p, p') &= 4\pi i \frac{4}{3} \alpha_s (\delta_\mu^\alpha Q_\nu - \delta_\nu^\alpha Q_\mu) p'_\beta \hat{D}_{\alpha\beta}(Q) - \\ &- \sigma \int d^3 \zeta e^{-i\mathbf{Q} \cdot \zeta} \varepsilon(p_0) \frac{\zeta_\mu p_\nu - \zeta_\nu p_\mu}{|\zeta| \sqrt{p_0^2 - \mathbf{p}_T^2}} p'_0 \end{aligned}$$

² In terms of the second order field $\phi(x) = (i\gamma^\mu D_\mu + m)^{-1} \psi(x)$ this wave function is defined by

$$\langle 0 | \phi(\frac{\xi}{2}) \bar{\psi}(-\frac{\xi}{2}) | P \rangle = \frac{1}{(2\pi)^2} \Phi_P(k) e^{-ik\xi}.$$

$$\begin{aligned}
\hat{I}_{0;\rho\sigma}(Q; p, p') &= -4\pi i \frac{4}{3} \alpha_s p^\alpha (\delta_\rho^\beta Q_\sigma - \delta_\sigma^\beta Q_\rho) \hat{D}_{\alpha\beta}(Q) + \\
&\quad + \sigma \int d^3 \zeta e^{-i\mathbf{Q} \cdot \zeta} p_0 \frac{\zeta_\rho p'_\sigma - \zeta_\sigma p'_\rho}{|\zeta| \sqrt{p_0'^2 - \mathbf{p}_T'^2}} \varepsilon(p'_0) \\
\hat{I}_{\mu\nu;\rho\sigma}(Q; p, p') &= \pi \frac{4}{3} \alpha_s (\delta_\mu^\alpha Q_\nu - \delta_\nu^\alpha Q_\mu) (\delta_\rho^\alpha Q_\sigma - \delta_\sigma^\alpha Q_\rho) \hat{D}_{\alpha\beta}(Q), \tag{8}
\end{aligned}$$

where in the second and in the third equation $\zeta_0 = 0$ has to be understood. Notice that, due to the privileged role given to the c. m. frame, the terms proportional to σ in (8) are not formally covariant.

3. DS EQUATION AND MASS OPERATOR

Concerning eq. (7), let us observe that the unity matrix, $\sigma^{\mu\nu}$ and γ^5 form a subalgebra of the Dirac algebra. Consequently $\Gamma(k)$ can be assumed to depend only on this set of matrices and, since it must be a three dimensional scalar, only on terms like $k_j \sigma^{0j}$. In fact, it can be checked that $\Gamma(k)$ can be consistently assumed to be completely spin independent and eq. (7) can be written in the form

$$\Gamma(k) = i \int \frac{d^4 l}{(2\pi)^4} \frac{R(k, l)}{l^2 - m^2 + \Gamma(l)}, \tag{9}$$

with

$$\begin{aligned}
R(k, l) &= 4\pi \frac{4}{3} \alpha_s \left[4 \frac{p^2 l^2 - (pl)^2}{(k-l)^2} + \frac{3}{4} \right] + \\
&\quad + \sigma \int d^3 \mathbf{r} e^{-i(\mathbf{k}-\mathbf{l}) \cdot \mathbf{r}} r (k_0 + l_0)^2 \sqrt{1 - \frac{(\mathbf{k}_\perp + \mathbf{l}_\perp)^2}{(k_0 + l_0)^2}}, \tag{10}
\end{aligned}$$

\mathbf{k}_\perp and \mathbf{l}_\perp denoting as above the transversal part of \mathbf{k} and \mathbf{l} .

Notice that, once (10) is solved, the pole or constituent mass m_P , to be used in bound states problems, is given by the equation

$$m_P^2 - m^2 + \Gamma(m_P^2) = 0. \tag{11}$$

We can try to solve eq. (9) iteratively and we have at the first step

$$\Gamma(k) = i \int \frac{d^4 l}{(2\pi)^4} \frac{R(k, l)}{l^2 - m^2}. \tag{12}$$

In a preliminary calculation we omit altogether the perturbative contribution to $R(k, l)$ (notice the overlapping of curves b and c in Fig. 1) and neglect the term in $(\mathbf{k}_\perp + \mathbf{l}_\perp)^2$ in the string part. Strictly, the second approximation is justified only for S bound states (classically $k_\perp r$ is the angular momentum of the bound state) but it is necessary in order to make the integral analytically calculable.

Then introducing a cut off μ , we obtain

$$\Gamma(k) = \frac{\sigma}{\pi} [k_0^2 A(m, |\mathbf{k}|) - B_\mu - B(m, |\mathbf{k}|)], \quad (13)$$

where $B_\mu = 2 \ln \frac{\mu}{m} - 1$,

$$A(m, |\mathbf{k}|) = \frac{1}{\mathbf{k}^2 + m^2} \left[1 + \frac{m^2}{2|\mathbf{k}|\sqrt{\mathbf{k}^2 + m^2}} \ln \frac{\sqrt{\mathbf{k}^2 + m^2} + |\mathbf{k}|}{\sqrt{\mathbf{k}^2 + m^2} - |\mathbf{k}|} \right] \quad (14)$$

and $B(m, |\mathbf{k}|)$ is a more complicated expression that we do not report explicitly here for lack of space. The resulting pole mass is

$$\bar{m}_P^2(m, |\mathbf{k}|) = \frac{m^2 + \frac{\sigma}{\pi} [B_\mu + B(m, |\mathbf{k}|) - \mathbf{k}^2 A(m, |\mathbf{k}|)]}{1 + \frac{\sigma}{\pi} A(m, |\mathbf{k}|)}, \quad (15)$$

The above expression depends on the current mass m and on the quark c. m. momentum $|\mathbf{k}|$, (see Fig. 1a). Notice that such dependence on $|\mathbf{k}|$ is clearly an artifact of the schematic way we have introduced confinement in eq. (5) and that the curve is rather flat in the region of interest. Correspondingly it seems reasonable to chose as true mass $m_P(m)$ the value of $\bar{m}_P(m, |\mathbf{k}|)$ at its stationary point in $|\mathbf{k}|$.

Then, in a neighbor of its singularity $k^2 = m_P^2$, the full propagator can be written as $\hat{H}^{(2)}(k) = \frac{iZ}{k^2 - m_P^2}$, where the residuum Z differs from 1 only for terms proportional to α_s or σ . Consistently in (6) we can simply take $Z = 1$ and are left with the free propagator with a constituent mass. If, in addition, we replace \hat{I}_{ab} with its so called instantaneous approximation $\hat{I}_{ab}^{\text{inst}}(\mathbf{k}, \mathbf{u})$, we can explicitly perform the integration in u_0 and arrive at a three dimensional reduced equation.

Such a reduced equation takes the form of the eigenvalue equation for a squared mass operator [3], $M^2 = M_0^2 + U$, with $M_0 = w_1 + w_2$, $w_{1,2} = \sqrt{m_{1,2}^2 + \mathbf{k}^2}$ and

$$\langle \mathbf{k} | U | \mathbf{k}' \rangle = \frac{1}{(2\pi)^3} \sqrt{\frac{w_1 + w_2}{2w_1 w_2}} \hat{I}_{ab}^{\text{inst}}(\mathbf{k}, \mathbf{k}') \sqrt{\frac{w'_1 + w'_2}{2w'_1 w'_2}} \sigma_1^a \sigma_2^b \quad (16)$$

(for an explicit expression we refer to [2, 1]). The quadratic form of the above equation obviously derives from the second order formalism we have used.

Alternatively, in more usual terms, one can look for the eigenvalue of the mass operator or center of mass Hamiltonian $H_{\text{CM}} \equiv M = M_0 + V$ with V defined by $M_0 V + V M_0 + V^2 = U$. Neglecting the term V^2 the linear form potential V can be obtained from U by the kinematic replacement $\sqrt{\frac{(w_1 + w_2)(w'_1 + w'_2)}{w_1 w_2 w'_1 w'_2}} \rightarrow \frac{1}{2\sqrt{w_1 w_2 w'_1 w'_2}}$. The resulting expression is particularly useful for a comparison with models based on potential. In particular, in the static limit V reduces to the Cornell potential

$$V_{\text{stat}} = -\frac{4}{3} \frac{\alpha_s}{r} + \sigma r; \quad (17)$$

in the semirelativistic limit (up to $\frac{1}{m^2}$ terms after an appropriate Foldy-Wouthuysen transformation) it equals the potential discussed in ref. [8], if full relativistic kinematics is kept, but the spin dependent terms are neglected, it becomes identical to the potential of the relativistic flux tube model [3].

4. RUNNING COUPLING CONSTANT AND MASSES

As we said, diagonalizing M^2 or H_{CM} with fixed coupling constant and quark masses, a general good fit of the data was obtained. Actually a serious problem was represented by the masses of the light pseudo scalar mesons that turned out too large. The results obtained in ref. [2] suggest, however, that the situation can be greatly improved using an appropriate running coupling constant and running quark masses.

At one loop, the running coupling constant is usually written

$$\alpha_s(Q) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)}, \quad (18)$$

with $\beta_0 = 11 - \frac{2}{3}N_f$ and N_f the number of ‘active’ quarks. However, the singularity occurring in such expression is an artifact of perturbation theory and it contradicts general analyticity properties, therefore the expression must be somewhat modified in the infrared region [9]. Notice that this is particularly important for the quark-antiquark bound state problem, where the variable Q^2 is usually identified with the squared momentum transfer $\mathbf{Q}^2 = (\mathbf{k} - \mathbf{k}')^2$, which ranges typically from $(0.1 \text{ GeV})^2$ to $(1 \text{ GeV})^2$ for different quark masses and states.

The most naive modification of eq. (18) would consist in freezing $\alpha_s(Q^2)$ to a certain maximum value $\bar{\alpha}_s$ as Q^2 decreases and in treating this value as a phenomenological parameter (truncation prescription). However, various more sensible proposals have been made on different bases [5, 6].

In particular Shirkov and Solovtsov [6] suggest to replace (18) with

$$\alpha_s(Q) = \frac{4\pi}{\beta_0} \left(\frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right). \quad (19)$$

This remains regular for $Q^2 = \Lambda^2$ and has a finite Λ independent limit $\alpha_s(0) = 4\pi/\beta_0$ for $Q^2 \rightarrow 0$. Eq. (19) is obtained assuming a dispersion relation for $\alpha_s(Q)$ with a cut for $-\infty < Q^2 < 0$ and applying (18) to the evaluation of the spectral function.

The running mass expression corresponding to (19) can be written in the form

$$m(Q) = \hat{m} \left(\frac{Q^2/\Lambda^2 - 1}{Q^2/\Lambda^2 \ln(Q^2/\Lambda^2)} \right)^{\gamma_0/2\beta_0}, \quad (20)$$

where in the \overline{MS} scheme $\gamma_0 = 8$. Eq. (20) is obtained integrating the one loop renormalization group equation

$$\frac{Q}{m(Q)} \frac{dm(Q)}{dQ} = -\gamma_0 \frac{\alpha_s(Q)}{4\pi}, \quad (21)$$

where (19) has been used and \hat{m} denotes an integration constant. Notice that $m(Q)$ is singular for $Q \rightarrow 0$, contrary to $\alpha_s(Q)$.

Finally, if we replace the running mass (20) in (15) after maximizing we obtain a running constituent mass $m_P(Q)$ of the type reported in Fig. 1 *d* that can be used together with the running coupling constant (19) in the expression of the operator M^2 (see Sec. 3).³

5. CALCULATIONS AND RESULTS

The calculations we report in this paper follow a similar line to those of Ref. [1]. The general strategy for solving the eigenvalue equation for M^2 and the numerical treatment are basically the same.

We neglect spin-orbit terms, but include the hyperfine terms in U (see eq. (16)). We solve first the eigenvalue equation for $M_{\text{stat}} = M_0 + V_{\text{stat}}$ (see Eq. (17)) by the Rayleigh-Ritz method with an harmonic oscillator basis and then treat $M^2 - M_{\text{stat}}^2$ as a perturbation (up to the first order this is obviously equivalent to taking $m_B^2 = \langle M^2 \rangle$).

In the above general framework, in Fig. 2 we graphically report and compare with the data [10] three different type of results, corresponding to different choices for the strong coupling constant α_s , the string tension σ and the constituent masses.

Diamonds correspond to results already reported in [1]. A running coupling constant $\alpha_s(Q)$ was assumed equal to the one loop perturbative expression (18) frozen at the maximum value $\bar{\alpha}_s = 0.35$, with $N_f = 4$ and $\Lambda = 200$ MeV. Q was identified with $|\mathbf{k} - \mathbf{k}'|$ and σ was set equal to 0.2 GeV^2 . Fixed masses $m_u = m_d = 10 \text{ MeV}$, $m_s = 200 \text{ MeV}$, $m_c = 1.394 \text{ GeV}$, $m_b = 4.763 \text{ GeV}$ were adopted. The results do not differ essentially from the fixed coupling constant case; the spectrum is reasonably well reproduced on the whole with the exception of the light pseudoscalar mesons π , η_s and K (the η_s mass is derived from the masses of η and η' with the usual assumptions).

Circlets correspond to results of the type reported in [2], but in which the hyperfine separation for the 1S and 2S states has been evaluated up to the second order of perturbation theory. In this case as running coupling constant we have taken the Shirkov-Solovtsov expression (19). We have set again $N_f = 4$ and $\Lambda = 200 \text{ MeV}$, but $\sigma = 0.18 \text{ GeV}^2$ and $m_s = 0.39 \text{ GeV}$, $m_c = 1.545 \text{ GeV}$, $m_b = 4.898 \text{ GeV}$. On the contrary for the light quarks we have taken a phenomenological running mass in terms of the modulus of the c.m. quark momentum, $m_u^2 = m_d^2 = 0.17|\mathbf{k}| - 0.025|\mathbf{k}|^2 + 0.15|\mathbf{k}|^4 \text{ GeV}^2$ with $|\mathbf{k}|$ in GeV. We can see that in this way even the light pseudoscalar mesons turn out correctly, with possibly some problems for the $q\bar{b}$ states and some other highly excited states for which coupling with other channels are probably important.

Squares correspond to preliminary completely new calculations, made using the analytic running constant (19), running constituent masses $m_P(Q)$ (as described in Sec.

³ At first sight it could seem strange that we should talk of a Q dependence for a quantity like the constituent mass that should have a definite physical value. The point is that we are using $m_P(Q)$ in the context of certain approximations and it is the accuracy of such approximations that depend on the scale Q .

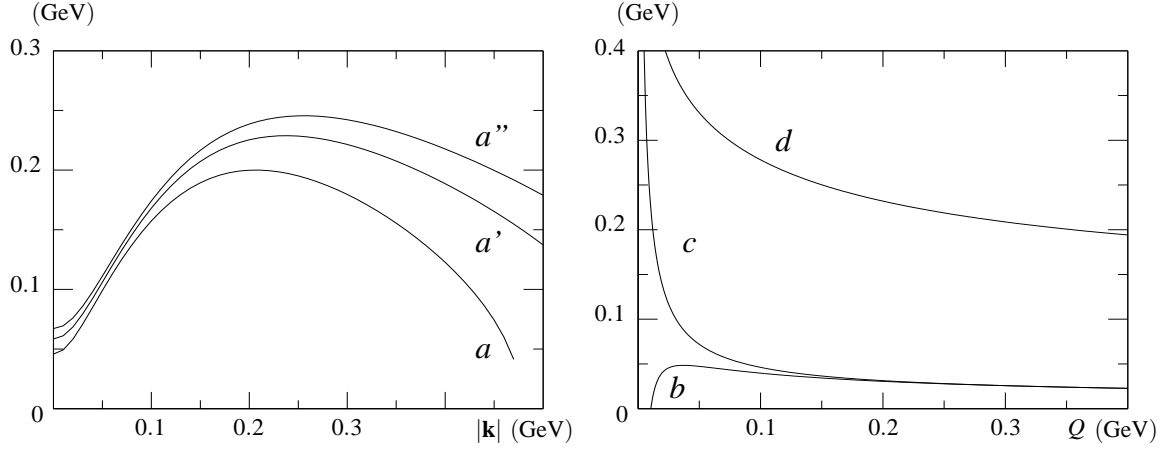


FIGURE 1. *a), a'), a'')* $\bar{m}_P = \bar{m}_P(m, |\mathbf{k}|)$ for increasing value of m (16); *d)* running constituent mass $m_P(Q)$; *c)* analytic running current mass (21); *b)* perturbative running pole masses corresponding to *c)*.

4) for the light and strange quarks, fixed masses for the charm and the beauty quarks. Inside $\alpha_s(Q)$ the quantity Q has been again identified with $|\mathbf{k} - \mathbf{k}'|$. On the contrary, for computational difficulties, inside $m_P(Q)$ we have taken

$$Q = \frac{1}{e^{\gamma_E \langle r \rangle}}, \quad (22)$$

where γ_E is the Euler constant $\gamma_E = 0.5772\dots$ and $\langle r \rangle$ is the radius of the unperturbed bound state [11]. We have chosen $N_f = 3$, $\Lambda = 180 \text{ MeV}$, $\sigma = 0.18 \text{ GeV}^2$, $\frac{\sigma}{\pi} B_\mu = 0.48 \text{ GeV}$ in (15), both for the light and the strange quarks, and then $\hat{m}_u = \hat{m}_d = 25.0 \text{ MeV}$, $\hat{m}_s = 87.3 \text{ MeV}$ in (20) (in order to reproduce correctly the ρ and the ϕ masses). Finally we have taken $m_c = 1.508 \text{ GeV}$ and $m_b = 4.842$ for c and b quarks. The results are not of a better quality than those obtained in the preceding calculation but obviously conceptually more satisfactory.

As an example, in the table numerical values for the three types of calculations are reported in the order for the light-light channel. For the third case in the last column the pertinent values of running constituent light quark mass are also reported for the various states.

6. CONCLUSIONS

In conclusion we can confirm what already noticed in references [2] that our reduced second order formalism together with ansatz (5) can reproduce reasonably well the general structure of the entire meson spectrum, light-light, heavy-heavy and light-heavy sectors included. In order to obtain the masses of light pseudo scalar mesons π , η_s and K , however, a correct consideration of the infrared behavior of the running coupling constant and of some kinds of running constituent mass for the light quarks is essential. The analytic Shirkov-Solovtsov coupling constant seems to provide such a behavior.

What is new in this paper is the inclusion of second order perturbative corrections to the hyperfine splitting in the case of phenomenological running masses considered in [2]

(circlets in Fig. 2) and the use of a running mass obtained combining renormalization group and analyticity requirements with an approximate solution of the quark Dyson-Schwinger equation (squares in Fig. 2).

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